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Adaptive Stabilization of Nonlinear Stochastic Systems

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Abstract. The purpose of this paper is to study the problem of asymptotic stabilization in probability of nonlinear stochastic differential systems with unknown parameters. With this aim, we introduce the concept of an adaptive control Lyapunov function for stochastic systems and we use the stochastic version of Artstein's theorem to design an adaptive stabilizer. In this framework the problem of adaptive stabilization of a nonlinear stochastic system is reduced to the problem of asymptotic stabilization in probability of a modified system. The design of an adaptive control Lyapunov function is illustrated by the example of adaptively quadratically stabilizable in probability stochastic differential systems.

Key Words. Stochastic differential equation, Asymptotic stability in probability, Adaptive stabilization, Adaptive control Lyapunov function.

AMS Classification. 60H10, 93C10, 93D05, 93D15, 93E15.

Introduction

The aim of this paper is to study the problem of asymptotic feedback stabilization in probability of nonlinear stochastic differential systems with an unknown constant parameter in the drift. Since in general this problem is not solvable by means of static feedback laws we introduce the concept of an adaptive control Lyapunov function and we use the stochastic version of Artstein's theorem established in [2] to design an adaptive stabilizer. In our framework the problem of adaptive stabilization of nonlinear stochastic differential systems is reduced to the problem of dynamic feedback stabilization, for all values of the unknown parameter, of a modified system.

The concept of a control Lyapunov function for stochastic differential systems has been introduced in [2] in order to prove a stochastic version of Artstein's theorem [1] (see also [9]). The asymptotic stability in probability of affine in the control nonlinear stochastic differential systems which can be characterized in terms of computable control Lyapunov functions which depend on the system coefficients has been studied in [4].

In [3] necessary and sufficient conditions for asymptotic feedback stability of stochastic differential systems are of Lyapunov-type. The stabilizers computed in this paper are smooth except possibly at their equilibrium state and their construction is based on the knowledge of a control Lyapunov function.

The concept of an adaptive control Lyapunov function associated with deterministic nonlinear systems with unknown parameters has been introduced by Krstić and Kokotović in [7]. In this paper we extend to nonlinear stochastic differential systems with unknown parameters the results proved for deterministic systems in [7]. The main tools used in the following are the stochastic Lyapunov machinery developed by Khasminskii [5] and the stochastic Artstein theorem established in [2].

This paper is divided into five sections organized as follows. In Section 1 we recall some definitions and results concerning the asymptotic stability in probability of the equilibrium solution of a stochastic differential equation proved by Khasminskii [5] and we recall the stochastic version of Artstein's theorem proved in [2]. In Section 2 we introduce the class of stochastic differential systems with unknown constant parameters we are dealing with in this paper. In Section 3 we state and prove the main result of the paper on the feedback stabilization of the class of stochastic differential systems introduced in the previous section. In Section 4 we illustrate the construction of an adaptive control Lyapunov function by proving a backstepping lemma and in Section 5 we design an example deduced from the "benchmark" model exposed in [6].

1. Stochastic Stability

The purpose of this section is to recall the main results concerning the asymptotic stability in probability of the equilibrium solution of a stochastic differential equation that we need in what follows as well as the stochastic version of Artstein's theorem. For a complete presentation of stochastic stability theory and stochastic Lyapunov machinery we refer the reader to the book by Khasminskii [5], for example.

1.1. *Asymptotic Stability in Probability*

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and denote by $w = \{w_t;\ t \geq 0\}$ a standard $\mathbb{R}^m$-valued Wiener process defined on this space.

Consider the stochastic process solution $x_t \in \mathbb{R}^n$ of the stochastic differential equation written in the sense of Itô:

$$x_t = x_0 + \int_0^t f(x_s)\, ds + \int_0^t g(x_s)\, dw_s, \tag{1}$$

where

1. x_0 is given in $\mathbb{R}^n$,
2. f and g are functionals mapping $\mathbb{R}^n$ into $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively, vanishing in the origin, and such that there exists a nonnegative constant K such that, for

any $x \in \mathbb{R}^n$,

$$|f(x)| + |g(x)| \leq K(1 + |x|).$$

If, for any $s \geq 0$ and $x \in \mathbb{R}^n$, $x_t^{s,x}$, $t \geq s$, denotes the solution at time t of the stochastic differential equation (1) starting from the state x at time s, the notion of asymptotic stability in probability for the equilibrium solution of the stochastic differential equation (1) can be introduced as follows.

Definition 1.1. The equilibrium solution $x_t \equiv 0$ of the stochastic differential equation (1) is **asymptotically stable in probability** if, and only if, for any $s \geq 0$ and $\varepsilon > 0$,

$$\lim_{x \to 0} P\left(\sup_{s \leq t} |x_t^{s,x}| > \varepsilon\right) = 0$$

and

$$P\left(\lim_{t \to +\infty} |x_t^{s,x}| = 0\right) = 1$$

for any $x \in \mathbb{R}^n$.

Denote by L the infinitesimal generator of the stochastic process solution x_t of the stochastic differential equation (1), that is, L is the second-order differential operator defined for any function Ψ in $C^2(\mathbb{R}^n; \mathbb{R})$ by

$$L\Psi(x) = \sum_{i=1}^{n} f^i(x) \frac{\partial \Psi}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 \Psi}{\partial x_i \partial x_j}(x), \tag{2}$$

where $a_{ij}(x) = \sum_{k=1}^{m} g_k^i(x) g_k^j(x)$, $1 \leq i, j \leq n$.

Then, by means of martingale theory arguments, the following version of the Lyapunov theorem can be proved.

Theorem 1.2 (see [5]). *Assume that there exists a Lyapunov function V defined on $\mathbb{R}^n$ (i.e., a functional V in $C^2(\mathbb{R}^n; \mathbb{R})$ which is proper and positive definite) such that*

$$LV(x) < 0$$

for any $x \in \mathbb{R}^n \backslash \{0\}$. Then the equilibrium solution $x_t \equiv 0$ of the stochastic differential equation (1) is asymptotically stable in probability.

For a detailed proof of Theorem 1.2 we refer the reader to Chapter V, pp. 156–171, of [5].

1.2. *The Stochastic Artstein Theorem*

Consider the stochastic process solution $x_t \in \mathbb{R}^n$ of the multi-input stochastic differential system written in the sense of Itô:

$$x_t = x_0 + \int_0^t (f(x_s) + h(x_s)u)\, ds + \int_0^t g(x_s)\, dw_s, \tag{3}$$

where

1. x_0 is given in $\mathbb{R}^n$,
2. u is a measurable $\mathbb{R}^p$-valued control law,
3. f and g are functionals defined as in the previous section,
4. h is a functional mapping $\mathbb{R}^n$ into $\mathbb{R}^{n\times p}$, vanishing in the origin, and such that, for any $x \in \mathbb{R}^n$,

   $|h(x)| \leq K(1 + |x|)$.

The stochastic differential system (3) is **asymptotically stabilizable in probability** if there exists a function k mapping $\mathbb{R}^n$ in $\mathbb{R}^p$, vanishing in the origin, and such that the equilibrium solution $x_t \equiv 0$ of the closed-loop system

$$x_t = x + \int_0^t (f(x_s) + h(x_s)k(x_s))\, ds + \int_0^t g(x_s)\, dw_s \tag{4}$$

is asymptotically stable in probability.

Then the concept of a control Lyapunov function for the stochastic differential system (3) can be introduced as follows.

Definition 1.3. A Lyapunov function V defined on $\mathbb{R}^n$ is said to be a **control Lyapunov function** for the stochastic differential system (3) if for every x in $\mathbb{R}^n\backslash\{0\}$ the following condition holds:

$$\sum_{j=0}^{n} h_i^j(x)\frac{\partial V}{\partial x_j}(x) = 0, \qquad i = 1, \ldots, p \quad \Rightarrow \quad LV(x) < 0.$$

A control Lyapunov function V associated with the stochastic differential system (3) is said to satisfy the **small control property** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathbb{R}^n\backslash\{0\}$ satisfies $||x|| < \delta$, then there exists a control u in $\mathbb{R}^p$ with $||u|| < \varepsilon$ such that

$$LV(x) + \sum_{i=1}^{p}\sum_{j=1}^{n} h_i^j(x)\frac{\partial V}{\partial x_j}(x)u^i < 0. \tag{5}$$

Then the stochastic version of Artstein's theorem asserts the following result.

Theorem 1.4 (see [2]). *If V is a control Lyapunov function for the stochastic differential system* (3) *which satisfies the small control property, then the state feedback law u defined on $\mathbb{R}^n$ by*

$$u(x) = \kappa(LV(x), \nabla V(x)h(x))(\nabla V(x)h(x))^\star, \tag{6}$$

where, for any (a, b) *in* $\mathbb{R}^2$,

$$\kappa(a,b) = \begin{cases} -\dfrac{a+\sqrt{a^2+b^2}}{b(1+\sqrt{1+b})} & \textit{if} \quad b>0, \\ 0 & \textit{if} \quad b=0, \end{cases} \tag{7}$$

renders the stochastic differential system (3) *asymptotically stable in probability.*

2. Setting of the Problem

The purpose of this section is to introduce the class of control stochastic differential systems with unknown constant parameter we are dealing with in this paper.

Denote by $x_t \in \mathbb{R}^n$ the stochastic process solution of the stochastic differential system written in the sense of Itô:

$$x_t = x_0 + \int_0^t (f(x_s) + F(x_s)\theta + h(x_s)u)\, ds + \int_0^t g(x_s)\, dw_s, \tag{8}$$

where

1. x_0 is given in $\mathbb{R}^n$,
2. u is a measurable real-valued control law,
3. θ is a constant **unknown** parameter with values in $\mathbb{R}^q$,
4. f, g, and h are functionals satisfying the hypothesis given in the previous section,
5. F is a function mapping $\mathbb{R}^n$ in $\mathbb{R}^{n\times q}$, vanishing in the origin, and such that, for every $x \in \mathbb{R}^n$,

   $$|F(x)| \leq K(1+|x|).$$

The stochastic differential system (8) is said to be **adaptively stabilizable in probability** if there exists a function $\alpha(x, \hat{\theta})$ smooth on $(\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^q$ with $\alpha(0, \hat{\theta}) \equiv 0$, a smooth functional $\tau(x, \hat{\theta})$, and a positive definite symmetric matrix Γ in $\mathcal{M}_{q\times q}(\mathbb{R})$ such that the dynamic control law

$$u = \alpha(x, \hat{\theta}), \tag{9}$$

$$\dot{\hat{\theta}} = \Gamma\ \tau(x, \hat{\theta}) \tag{10}$$

renders the equilibrium solution of the stochatic differential system (3) asymptotically stable in probability for any value of the parameter $\theta \in \mathbb{R}^q$.

In the following we substitute the problem of the adaptive stabilization of (8) by a problem of asymptotic feedback stabilization in probability of a modified system. With this aim, we introduce the notion of an adaptive control Lyapunov function as follows.

Definition 2.1. A function V_a in $C^2(\mathbb{R}^n \times \mathbb{R}^q, \mathbb{R})$, positive definite and proper in x for each value of θ, is said to be an **adaptive control Lyapunov function** for the stochastic differential system (8) if there exists a positive definite symmetric matrix Γ in $\mathcal{M}_{q\times q}(\mathbb{R})$

such that, for every θ in $\mathbb{R}^q$, $V_a(x,\theta)$ is a control Lyapunov function for the stochastic differential system

$$x_t = x_0 + \int_0^t \left(f(x_s) + F(x_s)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial\theta}(x_s,\theta)\right)^\star\right) + h(x_s)u\right) ds + \int_0^t g(x_s)\, dw_s. \tag{11}$$

3. Adaptive Stabilization in Probability

The purpose of this section is to design an adaptive stabilizing feedback law for the stochastic differential system (8) when an adaptive control Lyapunov function is known.

Theorem 3.1. *The following two assertions are equivalent*:

(1) *There exists a triple* (α, V_a, Γ) *such that* $u = \alpha(x,\theta)$ *asymptotically stabilizes in probability the stochastic differential system* (11) *for every* θ *in* $\mathbb{R}^q$ *with the Lyapunov function* $V_a(x,\theta)$.
(2) *There exists an adaptive control Lyapunov function* $V_a(x,\theta)$ *for the stochastic differential system* (8).

Proof of Theorem 3.1. (1) Since $u = \alpha(x,\theta)$ asymptotically stabilizes in probability the stochastic differential system (11), there exists a function W in $C(\mathbb{R}^n \times \mathbb{R}^q, \mathbb{R})$, positive definite in x for every $\theta \in \mathbb{R}^q$, such that

$$\mathcal{L}_\theta V_a(x,\theta) \le -W(x,\theta), \tag{12}$$

where $\mathcal{L}_\theta$ is the infinitesimal generator of the closed-loop system deduced from (11) when $u = \alpha(x,\theta)$.

Then $V_a(x,\theta)$ is a control Lyapunov function for the stochastic differential system (11) for every $\theta \in \mathbb{R}^q$, and, consequently, it is an adaptive control Lyapunov function for the stochastic differential system (8).

(2) If $V_a(x,\theta)$ is an adaptive control Lyapunov function for the stochastic differential system (8) it is, according to Definition 2.1, a control Lyapunov function for the stochastic differential system (11).

Therefore, the control law

$$\alpha(x,\theta) = \kappa(L_\theta V_a(x,\theta), (\nabla_x V_a(x,\theta)h(x))^2)\nabla_x V_a(x,\theta)h(x), \tag{13}$$

where κ is given by (7) and L_θ, the infinitesimal generator of the uncontrolled part of the stochastic differential system (11), is, according to Theorem 1.4, a smooth control law on $(\mathbb{R}^n\backslash\{0\}) \times \mathbb{R}$ which renders the equilibrium solution of the stochastic differential system (11) asymptotically stable in probability for every $\theta \in \mathbb{R}^q$.

Furthermore, note that the feedback law $u = \alpha(x,\theta)$ given by (13) will be continuous at $x = 0$ if, and only if, the adaptive control Lyapunov function V_a satisfies the small control property.

This completes the proof of Theorem 3.1. □

Proposition 3.2. *If there exists an adaptive control Lyapunov function for the stochastic differential system* (8)*, then this stochastic differential system is adaptively stabilizable in probability.*

Proof of Proposition 3.2. Assume that there exists an adaptive control Lyapunov function for the stochastic differential system (8). Then, according to Theorem 3.1, there exists a triple (α, V_a, Γ) and a continuous function W which is positive definite in x for every $\theta \in \mathbb{R}^q$ such that

$$\mathcal{L}_\theta V_a(x,\theta) \le -W(x,\theta),$$

where $\mathcal{L}_\theta$ is the infinitesimal generator of the closed-loop system deduced from (11) when $u = \alpha(x,\theta)$.

Let V be the Lyapunov function defined on $\mathbb{R}^n \times \mathbb{R}^q$ by

$$V(x,\hat{\theta}) = V_a(x,\hat{\theta}) + \tfrac{1}{2}(\theta-\hat{\theta})^\star \Gamma^{-1}(\theta-\hat{\theta}). \tag{14}$$

Denoting by $\mathcal{L}$ the infinitesimal generator of the stochastic process $(x_t, \hat{\theta}_t)$, the solution of (8)–(10) yields

$$\begin{aligned}\mathcal{L}V(x,\hat{\theta}) = {} & \mathcal{L}_{\hat{\theta}} V_a(x,\hat{\theta}) - \frac{\partial V_a}{\partial\hat{\theta}}(x,\hat{\theta})\Gamma\left(\frac{\partial V_a}{\partial x}(x,\hat{\theta})F(x)\right)^\star + \frac{\partial V_a}{\partial\hat{\theta}}(x,\hat{\theta})\Gamma\tau(x,\hat{\theta}) \\ & + (\theta-\hat{\theta})^\star\left(\frac{\partial V_a}{\partial x}(x,\hat{\theta})F(x)\right)^\star - (\theta-\hat{\theta})^\star\tau(x,\hat{\theta}).\end{aligned} \tag{15}$$

Therefore, choosing

$$\tau(x,\hat{\theta}) = \left(\frac{\partial V_a}{\partial x}(x,\hat{\theta})F(x)\right)^\star, \tag{16}$$

one has

$$\mathcal{L}V(x,\hat{\theta}) \le -W(x,\hat{\theta})$$

for every $\theta \in \mathbb{R}^q$.

Hence, the equilibrium solution $(x,\hat{\theta}) \equiv (0,\theta)$ of the stochastic differential system (8)–(10) is stable in probability and by means of the stochastic version of La Salle's theorem (see [8]) one can prove that x_t tends in probability to zero (see the proof of Theorem 3.7 in [3], for example); that is, the equilibrium solution of the stochastic differential system (8) is adaptively stabilizable in probability.

This completes the proof of Proposition 3.2. □

The control law $u = \alpha(x,\theta)$ given by (13) renders the stochastic differential system (11) asymptotically stable in probability but it may not be a stabilizer for the original stochastic differential system (8). However, as is shown in the proof of Proposition 3.2, the feedback law $u = \alpha(x,\hat{\theta})$ given by (13) and the update law $\dot{\hat{\theta}} = \Gamma\tau(x,\hat{\theta})$ with (16) is an adaptive stabilizing feedback law for the stochastic differential system (8).

The Lyapunov function V defined by (14) used in the proof of Theorem 3.1 is quadratic in the parameter error $\theta - \hat{\theta}$. This form is suggested by the linear dependence of the stochastic differential system (8) on the parameter θ and the fact that θ cannot be used for feedback.

In the following we prove that the quadratic form of the Lyapunov function (14) is necessary and sufficient for the existence of an adaptive control Lyapunov function.

Definition 3.3. The stochastic differential system (8) is **adaptively quadratically stabilizable in probability** if it is adaptively stabilizable in probability and there exists a function V_a in $C^2(\mathbb{R}^n \times \mathbb{R}^q; \mathbb{R})$, positive definite and proper in x for each value of $\theta \in \mathbb{R}^q$, and a function W in $C(\mathbb{R}^n \times \mathbb{R}^q; \mathbb{R})$, positive definite in x for every $\theta \in \mathbb{R}^q$, such that, for every $\theta \in \mathbb{R}^q$,

$$\mathcal{L}V(x, \hat{\theta}) \leq -W(x, \hat{\theta}),$$

where $\mathcal{L}$ is the infinitesimal generator of the stochastic process solution of (8)–(10) and V is the Lyapunov function given by (14).

Then, the following result on adaptive quadratic stability in probability can be proved.

Corollary 3.4. *The stochastic differential system* (8) *is adaptively quadratically stabilizable in probability if, and only if, there exists an adaptive control Lyapunov function.*

Proof of Corollary 3.4. The necessary part of the result is contained in the proof of Proposition 3.2.

We assume, now, that the stochastic differential system (8) is adaptively quadratically stabilizable in probability and prove first that $\tau(x, \hat{\theta})$ must be given by (16).

For any $(x, \hat{\theta}) \in \mathbb{R}^n \times \mathbb{R}^q$, equality (15) can be rewritten as

$$\begin{aligned}\mathcal{L}V(x, \hat{\theta}) = {} & \mathcal{L}_{\hat{\theta}} V_a(x, \hat{\theta}) - \frac{\partial V_a}{\partial \hat{\theta}}(x, \hat{\theta})\Gamma \left(\frac{\partial V_a}{\partial x}(x, \hat{\theta})F(x)\right)^{\star} \\ & + \frac{\partial V_a}{\partial \hat{\theta}}(x, \hat{\theta})\Gamma\tau(x, \hat{\theta}) - \hat{\theta}^{\star}\left(\left(\frac{\partial V_a}{\partial x}(x, \hat{\theta})F(x)\right)^{\star} - \tau(x, \hat{\theta})\right) \\ & + \theta^{\star}\left(\left(\frac{\partial V_a}{\partial x}(x, \hat{\theta})F(x)\right)^{\star} - \tau(x, \hat{\theta})\right). \end{aligned} \tag{17}$$

Then, for the stochastic differential equation (8) to be adaptively quadratically stabilizable in probability, this expression has to be nonpositive. Therefore, since the right-hand side of equality (17) is affine in θ, it is nonpositive for every $x \in \mathbb{R}^n$ and $\theta, \hat{\theta} \in \mathbb{R}^q$ only if the last term is zero; that is, if

$$\tau(x, \hat{\theta}) = \left(\frac{\partial V_a}{\partial x}(x, \hat{\theta})F(x)\right)^{\star}.$$

Furthermore, in this case, it can be easily deduced from Theorem 3.1 that $V_a(x,\theta)$ is an adaptive control Lyapunov function for the stochastic differential equation (8).

This completes the proof of Corollary 3.4. □

4. A Backstepping Lemma

The purpose of this section is to extend the adaptive backstepping design with tuning function proved in [6] to the class of stochastic differential systems introduced in Section 2.

An adaptive control Lyapunov function for a higher-order system can be deduced by means of backstepping from an adaptive control Lyapunov function for a lower-order system.

Proposition 4.1. *If the stochastic differential system* (8) *is adaptively quadratically stabilizable in probability with* α *in* $C^2(\mathbb{R}^n \times \mathbb{R}^q; \mathbb{R})$, *then the augmented system*

$$\begin{cases} x_t = x_0 + \int_0^t (f(x_s) + F(x_s)\theta + h(x_s)\xi)\,ds + \int_0^t g(x_s)\,dw_s, \\ \dot{\xi} = u \end{cases} \tag{18}$$

is also adaptively quadratically stabilizable in probability.

Proof of Proposition 4.1. Since the stochastic differential system (8) is adaptively quadratically stabilizable in probability one can deduce from Corollary 3.4 that there exists an adaptive control Lyapunov function $V_a(x,\theta)$ which satisfies, according to Theorem 3.1, inequality (12) with a control law $u = \alpha(x,\theta)$.

In the following we prove that the Lyapunov function V_1, defined on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^q$ by

$$V_1(x,\xi,\theta) = V_a(x,\theta) + \tfrac{1}{4}(\xi - \alpha(x,\theta))^4, \tag{19}$$

is an adaptive control Lyapunov function for the stochastic differential system (18) by showing that it is a control Lyapunov function for the stochastic differential system

$$d\begin{pmatrix} x_t \\ \xi_t \end{pmatrix} = \begin{pmatrix} f(x_t) + F(x_t)(\theta + \Gamma((\partial V_1/\partial\theta)(x,\xi,\theta))^\star) + h(x_t)\xi_t \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u\,dt + \begin{pmatrix} g(x_t) \\ 0 \end{pmatrix} dw_t. \tag{20}$$

With (19) one has

$$\nabla V_1(x,\xi,\theta)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \quad \Rightarrow \quad \xi = \alpha(x,\theta).$$

On the other hand, denoting by $\mathcal{L}_1$ the infinitesimal generator of the uncontrolled part of the stochastic differential system (20) yields

$$\mathcal{L}_1 V_1(x,\xi,\theta)|_{\xi=\alpha(x,\theta)} = \mathcal{L}_\theta V_a(x,\theta),$$

where $\mathcal{L}_\theta$ is the infinitesimal generator of the closed-loop system deduced from (11) when $u = \alpha(x, \theta)$.

Therefore, since $V_a(x, \theta)$ is an adaptive control Lyapunov function for the stochastic differential system (8), one has

$$\mathcal{L}_\theta V_a(x, \theta) < 0$$

for every $\theta \in \mathbb{R}^q$.

Hence, according to Definition 1.3, V_1 is a control Lyapunov function for the stochastic differential system (20). Then, according to Theorem 3.1, V_1 is an adaptive control Lyapunov function for the stochastic system (18) and by Corollary 3.4 this system is adaptively quadratically stabilizable in probability.

This completes the proof of Proposition 4.1. □

The tuning function associated with the stochastic differential system (18) is defined for any $(x, \xi, \theta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^q$ by

$$\begin{aligned}\tau_1(x, \xi, \theta) &= \left(\frac{\partial V_1}{\partial(x, \xi)}(x, \xi, \theta) \begin{pmatrix} F(x) \\ 0 \end{pmatrix} \right)^\star \\ &= \left(\left(\frac{\partial V_a}{\partial x}(x, \theta) - (\xi - \alpha(x, \theta))^3 \frac{\partial \alpha}{\partial x}(x, \theta) \right) F(x) \right)^\star \\ &= \tau(x, \theta) - \left(\frac{\partial \alpha}{\partial x}(x, \theta) F(x) \right)^\star (\xi - \alpha(x, \theta))^3,\end{aligned}$$

where τ is the tuning function associated with the stochastic differential system (8).

5. A Design Example

Let x_0 be given in $\mathbb{R}^3$ and denote by $x_t \in \mathbb{R}^3$ the solution of the stochastic differential system written in the sense of Itô:

$$dx_t = \begin{pmatrix} x_{2,t} \\ x_{3,t} \\ 0 \end{pmatrix} dt + \begin{pmatrix} \varphi(x_{1,t}) \\ 0 \\ 0 \end{pmatrix} \theta \, dt + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \, dt \begin{pmatrix} 0 \\ 0 \\ x_{1,t} \end{pmatrix} dw_t, \tag{21}$$

where φ is a smooth functional defined on $\mathbb{R}$ and w is a standard real-valued Wiener process.

Note that (21) is deduced from the "benchmark" example exposed in [6] by adding a "noisy term" in the equation defining the third component of the system.

Introduce the stochastic process $z_t \in \mathbb{R}^3$ defined by

$$z_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} - \alpha_1(x_{1,t}, \hat{\theta}_t) \\ x_{3,t} - \alpha_2(x_{1,t}, x_{2,t}, \hat{\theta}_t) \end{pmatrix},$$

where the stabilizing functions α_1, α_2 and the state feedback control law are

$$\alpha_1(x_1,\hat{\theta}) = -\tfrac{3}{2}x_1 - \hat{\theta}\varphi(x_1),$$

$$\alpha_2(x_1,x_2,\hat{\theta}) = -z_2 - z_1 + \frac{\partial\alpha_1}{\partial x_1}.(x_2+\hat{\theta}\varphi) + \frac{\partial\alpha_1}{\partial\hat{\theta}}\tau_2 + \frac{1}{2}\frac{\partial^2\alpha_1}{\partial x_1^2}x_1^2,$$

$$\begin{aligned} u = {} & -z_2 - z_3 + \frac{\partial\alpha_2}{\partial x_1}.(x_2+\hat{\theta}\varphi) + \frac{\partial\alpha_2}{\partial x_2}x_3 + \frac{\partial\alpha_2}{\partial\hat{\theta}}\tau_3 \\ & -z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}\frac{\partial\alpha_2}{\partial x_1}\varphi + \frac{1}{2}\frac{\partial^2\alpha_1}{\partial x_1^2}x_1^2 + \frac{1}{2}\frac{\partial^2\alpha_2}{\partial x_2^2}x_1^2, \end{aligned}$$

and the tuning functions and the update law for $\hat{\theta}$ are given by

$$\tau_1 = z_1\varphi, \quad \tau_2 = \tau_1 - z_2\frac{\partial\alpha_1}{\partial x_1}\varphi, \quad \tau_3 = \tau_2 - z_3\frac{\partial\alpha_2}{\partial x_1}\varphi,$$

and

$$\frac{d\hat{\theta}}{dt} = \tau_3 = z_1\varphi - z_2\frac{\partial\alpha_1}{\partial x_1}\varphi - z_3\frac{\partial\alpha_2}{\partial x_1}\varphi.$$

Then, by using the Lyapunov function V defined on $\mathbb{R}^3 \times \mathbb{R}$ by

$$V(z,\theta) = \tfrac{1}{2}||z||^2 + \tfrac{1}{2}(\theta-\hat{\theta})^2,$$

one can prove easily that the equilibrium solution of the stochastic differential system

$$\begin{aligned} dz_t = {} & \begin{pmatrix} -\frac{3}{2}z_1 + z_2 \\ -z_1 - z_2 + z_3 + (\partial\alpha_1/\partial\hat{\theta})(\partial\alpha_2/\partial x_1)z_3\varphi(z_1) \\ -z_2 - z_3 - (\partial\alpha_1/\partial\hat{\theta})(\partial\alpha_2/\partial x_1)z_2\varphi(z_1) \end{pmatrix} dt \\ & + \begin{pmatrix} \varphi(z_1) \\ -(\partial\alpha_1/\partial x_1)\varphi(z_1) \\ -(\partial\alpha_2/\partial x_1)\varphi(z_1) \end{pmatrix} (\theta-\hat{\theta})\,dt + \begin{pmatrix} 0 \\ 0 \\ z_1 \end{pmatrix} dw_t \end{aligned}$$

is asymptotically stable in probability.

Therefore, the stochastic differential system (21) is adaptively asymptotically stabilizable in probability.

6. Conclusions

The problem of adaptive stabilization in probability is difficult because the functional V_a used in our framework, which modifies the stochastic differential system (3), has to be its own Lyapunov function.

The discussion developed above extends also for stochastic differential systems in the form

$$x_t = x_0 + \int_0^t (f(x_s) + F(x_s)\theta + (h(x_s) + H(x_s)\theta)\, u)\, ds + \int_0^t g(x_s)\, dw_s.$$

In this case, the existence of an adaptive control Lyapunov function V_a is equivalent to the existence of a control Lyapunov function for the stochastic differential system

$$\begin{aligned} x_t = x_0 + \int_0^t \Bigg(f(x_s) + F(x_s)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial\theta}(x_s,\theta)\right)^{\star}\right) \\ + \left(h(x_s) + H(x_s)\left[\theta + \Gamma\left(\frac{\partial V_a}{\partial\theta}(x_s,\theta)\right)^{\star}\right]\right) u \Bigg)\, ds \\ + \int_0^t g(x_s)\, dw_s. \end{aligned}$$

The extension to the case of multi-input stochastic differential systems is straightforward.

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